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The equations of the perturbed motion of a rocket as a thin-walled structure with a liquid☆

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ABSTRACT

The perturbed motion of a rocket as an elastic thin-walled structure with compartments partially filled with liquid propellant is considered. It is assumed that the normal modes of the hydroelastic oscillations of the rocket are determined under the condition that the velocity potential on the free surface of the liquid is equal to zero and with standard remaining conditions. Certain features of these modes with zero fundamental frequencies are pointed out and the "loss" of mass effect associated with this is explained. Equations are derived for the perturbed motion of a rocket taking account of the hydroelastic oscillations of its structure and the oscillations of the liquid with deviations of the free surface from the equilibrium position under the action of mass forces. The coefficients of these equations, characterizing the relation between the different type of oscillations, are expressed in terms of known hydrodynamic parameters and the values of the oscillation modes at certain points.

Simple models of an elastic rod with a mass and stiffness which are variable along its length are still used in solving problems in rocket dynamics. The liquid propellant is taken into account in these models as a linear mass. The effect of the oscillations, caused by deviations of the free surface of the liquid from the equilibrium position, are taken into account by introducing equivalent oscillators in certain sections of the rod. The longitudinal elastic oscillations are considered separately. The axi-symmetric oscillations of the tanks, as shells containing a liquid, play an important role for them. An elastic rod with oscillators in sections of the ribs of the tanks serves as a computational model in this case. Equations for the perturbed motion of the rocket using these models have been presented by Kolesnikov[1] and, in a somewhat different form, by Rabinovich.[2]

The investigations of Moiseyev and Rumyantsev[3] dealt with general theoretical problems of the dynamics of an elastic body with a liquid. Mathematical models of structures containing shells with a liquid, including taking account of its compressibility, were proposed by of Shklyarchuk.[4]

The increase in the memory and speed of computers has led to the creation by the finite element method of matrix models of structures with a liquid which increasingly reflect their dynamic properties. Here, as a rule, the condition that the velocity potential of the liquid must be equal to zero on the free surface is imposed. In the case of complex structures, matrix models of the individual compartments can be constructed and a synthesis can then be carried out using the Craig–Bampton method.[5] As a result, the situation arose that the contemporary mathematical models of structures with a liquid for the calculating the modes of elastic oscillations conflicted with the existing perturbed motion equations.[1,2]

The correct determination of the modes of the longitudinal elastic oscillations of rockets with a tandem configuration and all modes of oscillations of rockets with liquid strap-ons is only possible either under the condition that the velocity potential of the liquid is equal to zero or under the condition that the pressure on the free surface of the liquid is constant. If one uses the latter condition, the normal modes of the oscillations are determined taking account of the wave motions of the liquid caused by the action of mass forces, and the equations of perturbed motion take the simplest form. However, at the same time, a large number of modes appear in the low frequency region which have practically no effect either on the controlled motion of the rocket or on the dynamic loads, but make the calculations and the analysis of their results considerably more complicated.

The choice of the boundary condition on the free surface of the liquid is determined to a considerable extent by the fact that structure elements (antislosh baffles) for a strong increase in damping as the amplitude of the sloshing increases are often installed in the propellant

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tanks in order to ensure stability of the motion of the rocket. These amplitudes are simply expressed in terms of the generalized coordinates characterizing the sloshing in a fixed tank which necessitates their use irrespective of the other generalized coordinates. Otherwise, the damping factors will be non-linear functions of all the generalized coordinates.

In this paper, the equation for the perturbed motion of a rocket are obtained for use in cases when its dynamic characteristics as an elastic sturcture with a liquid is synthesized using modern methods. These of the condition that the velocity potential on the free surface of the liquid is equal to zero is recommended when calculating the modes of the hydroelastic oscillations of the rocket structure or its separate compartments by the finite element method. It is important that, in this case, there is no need to know the flight cyclogram and the overloads which arise. Furthermore, the sloshing frequencies that have a significant effect on the motion of the rocket found to be are noticeably lower than the frequencies of the elastic oscillations of the structure. Hence, the modes of the elastic oscillations, obtained under the above-mentioned condition, are closest to the real normal modes of the elastic oscillations of a rocket.

1. The velocity potential of the liquid when there are elastic oscillations of the structure

Suppose the normal modes of elastic oscillations of the structure of a rocket are determined when the velocity potential on the free surface of the liquid Σ is equal to zero. Among these modes, there are also six modes with zero natural frequencies We will denote the vectors of the displacement of the centre of mass of the rocket and its rotation around the centre of mass by $\mathbf{u}$ and $\boldsymbol{\theta}$. The vector of the generalized coordinates of the elastic oscillations is denoted by $\mathbf{q}$.

Assuming that the oscillations are small, using the principle of superposition the velocity potential of the liquid in a rocket tank is represented in the form

$$\chi^\circ(t,\mathbf{r}) = \dot{\mathbf{u}}(t)\mathbf{R}^\circ(\mathbf{r}) + \dot{\theta}(t)\Psi^\circ(\mathbf{r}) + \dot{\mathbf{q}}(t)\Phi^\circ(\mathbf{r}) \tag{1.1}$$

The vector functions $\mathbf{R}^\circ,\Psi^\circ,\Phi^\circ$ satisfy Laplace's equation in the bulk of the liquid V, the impermeability condition on the walls of the tank S and $\mathbf{R}^\circ = 0, \Psi^\circ = 0, \Phi^\circ = 0$ on the free surface of the liquid.

The functions $\mathbf{R}^\circ$ and Ψ° can be expressed in terms of the eigenfunctions ϕ_n of the boundary-value problem of the sloshing[4]:

$$\Delta\varphi_n = 0; \quad \partial\varphi_n/\partial\nu\big|_S = 0, \quad \partial\varphi_n/\partial\nu\big|_\Sigma = \kappa_n\varphi_n\big|_\Sigma \tag{1.2}$$

and the Stokes–Zhukovskii potentials of the Neumann problem:

$$\Delta\Psi = 0; \quad \partial\Psi/\partial\nu\big|_{S\cup\Sigma} = \mathbf{r}\times\boldsymbol{\nu} \tag{1.3}$$

where $\boldsymbol{\nu}$ is a unit vector of the outword normal. Representing the function $\mathbf{R}^\circ$ and Ψ° in the form

$$\mathbf{R}^\circ = \mathbf{r} - \sum_n \mathbf{a}_n\varphi_n, \quad \Psi^\circ = \Psi - \sum_n \mathbf{b}_n\varphi_n \tag{1.4}$$

multiplying these expressions by $\partial\varphi_k/\partial\nu$ and integrating over the free surface of the liquid, we find

$$\mathbf{a}_n = \lambda_n/\mu_n, \quad \mathbf{b}_n = \lambda_{On}/\mu_n \tag{1.5}$$

Here, the condition that the functions φ_n are orthogonal has been used and the notation for the known hydrodynamic coefficients[6] is introduced

$$\lambda_n = \rho\int_\Sigma \mathbf{r}\frac{\partial\varphi_n}{\partial\nu}dS, \quad \lambda_{On} = \rho\int_\Sigma \Psi\frac{\partial\varphi_n}{\partial\nu}dS, \quad \mu_n = \rho\int_\Sigma \varphi_n\frac{\partial\varphi_n}{\partial\nu}dS \tag{1.6}$$

where ρ is the liquid density. If problem (1.2) has multiple eigenvalues κ_n, then the orthogonal eigenfunctions corresponding to them are selected.

We point out a feature of the modes of motion with zero natural frequencies. Substituting expressions (1.5) for the coefficients into formulae (1.4), we obtain

$$\rho\int\left[\nabla(\dot{\mathbf{u}}\mathbf{R}^\circ)\right]^2 dV = m\dot{\mathbf{u}}^2 - \sum_n \frac{(\dot{\mathbf{u}}\lambda_n)^2}{\mu_n}, \quad \rho\int\left[\nabla(\dot{\theta}\Psi^\circ)\right]^2 dV = \dot{\theta}J\dot{\theta} - \sum_n \frac{(\dot{\theta}\lambda_{On})^2}{\mu_n} \tag{1.7}$$

where m is the mass of the liquid in a tank and J is the inertia tensor of the liquid:

$$J_{ij} = \rho\int_V \nabla\Psi_i\nabla\Psi_j dV, \quad i,j = 1,2,3$$

Integration is carried out over the volume of the liquid.

It follows from expressions (1.7) that part of the mass of the liquid does not participate in the absolute motion but moves relative to the tank. This is due to the fact that, in the case of the boundary conditions on the free surface which have been adopted, the whole spectrum of natural sloshing frequencies is shifted to the zero point. Since the vectors λ_n are perpendicular to the direction of the apparent acceleration, the generalized mass is equal to the physical mass in this direction.

Note that the normal modes satisfy the orthogonality conditions

$$\int \dot{\mathbf{u}}\mathbf{w}_k dm + \rho\int \nabla(\dot{\mathbf{u}}\mathbf{r})\nabla\Phi_k^\circ dV = 0$$
$$\int \dot{\theta}\mathbf{w}_k dm + \rho\int_V \nabla(\dot{\theta}\Psi^\circ)\nabla\Phi_k^\circ dV = 0$$
$$\int \mathbf{w}_i\mathbf{w}_k dm + \rho\int \nabla\Phi_i^\circ\nabla\Phi_k^\circ dV = 0;\quad i \neq k \tag{1.8}$$

The integration in the first terms is carried out over all the mass elements of the structure dm, $\mathbf{w}_k$ is the mode of the elastic oscillations and Φ_k° is the component of the vector function $\Phi\circ : \mathbf{q}(t)\Phi\circ(\mathrm{r}) = \sum_k q_k(t)_k^\circ(r)$ corresponding to this mode.

2. The equation of the perturbed motion

We will consider the perturbed motion in the non-inertial bound system of coordinates for the unperturbed motion of a rocket. The origin of the system of coordinates is placed at the centre of mass of the structure with the liquid in its unperturbed state. By the unperturbed motion, we mean a motion during which the centre of mass moves with an acceleration which is constant in magnitude and direction, the angular velocities are equal to zero, there are no elastic oscillations of the structure, and the liquid is at rest with respect to the tank.

The displacement of the centre of the plane mirror of the free surface of the liquid and its rotation for the corresponding mode of elastic oscillations of the structure are denoted by $q_k(t)\mathbf{w}_k^\circ$ and $q_k(t)\theta_k^\circ$. We will determine the oscillations of the liquid caused by the displacement of its free surface relative to this mirror.

We now represent the velocity potential of the liquid in the tank in the form

$$\chi(t,\mathbf{r}) = \dot{\mathbf{u}}(t)\mathbf{r} + \dot{\theta}(t)\Psi(\mathbf{r}) + \sum_k \dot{q}_k(t)\Phi_k(\mathbf{r}) + \sum_n \dot{s}_n(t)\varphi_n(\mathbf{r}) \tag{2.1}$$

where S_n are the generalized coordinates of the oscillations of the free surface. If the treatment is restricted to the lowest fundamental modes of the elastic oscillations, then, in the free surface and on the tank walls close to it, the functions Φ_k must satisfy the boundary condition

$$\partial\Phi_k/\partial\nu \approx \mathbf{w}_k^\circ\boldsymbol{\nu} + (\theta_k^\circ \times \mathbf{r}^\circ)\boldsymbol{\nu} \tag{2.2}$$

where $\mathbf{r}^\circ$ is the radius vector of a point from the centre of the free surface. Hence, close to the free surface in the first approximation

$$\Phi_k(\mathbf{r}) \approx \mathbf{w}_k^\circ\mathbf{r} + \theta_k^\circ\Psi' \tag{2.3}$$

and, if relation (2.2) is taken into account, the Stokes–Zhukovskii potentials Ψ' of problem (1.3) are defined relative to the centre of the free surface.

We will obtain the equations of the perturbed motion by Lagrange's method. To do this, we write the quadratic forms of the kinetic energy of the liquid

$$T = \frac{1}{2}\rho\int(\nabla\chi)^2 dV \tag{2.4}$$

and the potential energy of the liquid

$$\Pi = \rho j\sum_n \int_\Sigma s_n\frac{\partial\varphi_n}{\partial\nu}\left[\left[\left(\theta + \sum_k q_k\theta_k^\circ\right)\times\mathbf{r}\right]\boldsymbol{\nu} + \frac{1}{2}s_n\frac{\partial\varphi_n}{\partial\nu}\right]dS \tag{2.5}$$

The velocity potential χ is defined by expression (2.1), and j is the acceleration of the mass forces in the non-inertial system of coordinates.

In order that the orthogonality conditions (1.9) can be used to determine the coefficients of the equations of the perturbed motion, it is necessary to express the functions Φ_k in terms of the functions Φ_k° of the potential (1.1). Since the sloshing of the liquid, due to the displacement of the free surface, decay quite rapidly on departing from it, these functions are only significantly different close to this surface and, therefore, in the first approximation

$$\Phi_k(\mathbf{r}) \approx \Phi_k^\circ(\mathbf{r}) + \sum_n a_{kn}\varphi_n(\mathbf{r}) \tag{2.6}$$

Multiplying relation (2.6) by $\partial\varphi_n/\partial\nu$ and integrating both sides of the resulting approximate equation over the free surface of the liquid, having first replaced $\Phi_k(\mathbf{r})$ in accordance with expression (2.3), we find

$$a_{kn} = (\mathbf{w}_k^\circ\lambda_n + \theta_k^\circ\lambda'_{On})/\mu_n \tag{2.7}$$

where λ_{On}' is determined using the second formula in (1.6) by replacing Ψ by Ψ'. Using Green's formula, it is easy to show that

$$\lambda'_{On} = \lambda_{On} - (\mathbf{r} - \mathbf{r}^\circ)\times\lambda_n \tag{2.8}$$

Here, $\mathbf{r} - \mathbf{r}^\circ$ is the constant radius vector from the origin of coordinates to the centre of the free surface of the liquid.

Using the expressions for the kinetic and potential energies of the liquid, we represent the Lagrange's equations in the form

$$
\begin{aligned}
&(m_0+m)\ddot{\mathbf{u}}+\sum_n \lambda_n \ddot{s}_n+\sum_k \mathbf{m}_{uk}\ddot{q}_k = \delta\mathbf{F}\\
&(J_0+J)\ddot{\theta}+\sum_n[\lambda_{On}\ddot{s}_n+(\mathbf{j}\times\lambda_n)s_n]+\sum_k \mathbf{m}_{\theta k}\ddot{q}_k = \delta\mathbf{M}_O\\
&\mu_n\left(\ddot{s}_n+\beta_n^s\dot{s}_n+\omega_n^2 s_n\right)+\lambda_n\ddot{\mathbf{u}}+\lambda_{On}\ddot{\theta}+(\mathbf{j}\times\lambda_n)\theta+\sum_k\left[\lambda_{nk}\ddot{q}_k+(\mathbf{j}\times\lambda_n)\theta_k^\circ q_k\right]=0\\
&\ddot{q}_k+\beta_k^q q_k+\sigma_k^2 q_k+\mathbf{m}_{uk}\ddot{\mathbf{u}}+\mathbf{m}_{\theta k}\ddot{\theta}+\sum_l m_{kl}\ddot{q}_l+\sum_n\left[\lambda_{nk}\ddot{s}_n+(\mathbf{j}\times\lambda_n)\theta_k^\circ s_n\right]=Q_k\\
&n=1,2,\ldots;\quad k,l=1,2,\ldots
\end{aligned}
\tag{2.9}
$$

In these equations, m_0 and J_0 are the mass and inertia tensor of the structure without the liquid in the tank, and $\delta\mathbf{F}$ and $\delta\mathbf{M}_O$ are the variations of the external forces and moments. It is assumed that the modes of elastic oscillations, the frequencies of which are denoted by σ_k, are normalized to unit generalized masses. Dissipative forces are formally taken into account by the diagonal matrices for the damping of the sloshing and the elastic oscillations of the structure. The sloshing frequencies are expressed in terms of the eigenvalues of problem (1.2): $\omega_n^2=j\kappa_n$.

In Eqs (2.9), the coefficients λ_{nk},$\mathbf{m}_{uk}$,$\mathbf{m}_{\theta k}$,$\mathbf{m}_{lk}$, characterizing the relation between the different forms of oscillations, are found using the formulae

$$
\lambda_{nk}=\mathbf{w}_k^\circ\lambda_n+\theta_k^\circ\lambda'_{On},\quad \mathbf{m}_{uk}=\sum_n\frac{\lambda_{nk}}{\mu_n}\lambda_n,\quad \mathbf{m}_{\theta k}=\sum_n\frac{\lambda_{nk}}{\mu_n}\lambda_{On},\quad m_{kl}=\sum_n\frac{\lambda_{nk}\lambda_{nl}}{\mu_n}
\tag{2.10}
$$

like the corresponding coefficients of the quadratic form of the kinetic energy of the liquid (2.4) with velocity potential (2.1). Representations (1.4) and (2.6), the expressions for the coefficients in these representations (1.5) and (2.7) and orthogonality conditions (1.9) have been used in obtaining these formulae. Summation on the right-hand sides of equalities (2.10) is carried out over all the sloshing modes considered.

For instance, in determining the coefficients $\mathbf{m}_{uk}$ and $\mathbf{m}_{\theta k}$, the corresponding integrals were represented in the form

$$
\int\nabla(\dot{\mathbf{u}}\mathbf{r})\nabla\Phi_k dV=\int\nabla\left(\dot{\mathbf{u}}\mathbf{R}^\circ+\sum_n\dot{\mathbf{u}}\mathbf{a}_n\varphi_n\right)\nabla\left(\Phi_k^\circ+\sum_n a_{kn}\varphi_n\right)dV
$$

$$
\int\nabla(\dot{\theta}\Psi)\nabla\Phi_k dV=\int\nabla\left(\dot{\theta}\Psi^\circ+\sum_n\dot{\mathbf{u}}\mathbf{b}_n\varphi_n\right)\nabla\left(\Phi_k^\circ+\sum_n a_{kn}\varphi_n\right)dV
$$

Account was taken of the fact that $\mathbf{R}^\circ=0$, $\Psi^\circ=0$, $\Phi^\circ=0$ on the free surface of the liquid Σ and Green's formula for harmonic functions was used.

If propellant is consumed from the tank, the coefficients of the quadratic form of the kinetic energy of the liquid (2.4) are time-dependent. However, the maximum period of the oscillatory processes in the perturbed motion is many times less than the time of flight of a stage of a rocket. The derivative of these coefficients with respect to time are therefore usually small, and the terms corresponding to them in Eqs (2.9) are not taken into account, which is not always admissible.

As a rule, the sloshing modes affecting the motion of a rocket can be subdivided into two types. For the first type $\lambda_{On}=\mathrm{r}_n\times\lambda_n$, where $\mathbf{r}_n$ is the radius vector from the centre of mass to the point of application of the principal vector of the hydrodynamic forces. It is then possible to change to new generalized coordinates of the sloshing using the formula $s_n=(\lambda_n/\mu_n)s'_n$. In the case of the second type $\lambda_i=0$ and it follows from formula (2.8) that $\lambda'_{Oi}=\lambda_{Oi}$, that is, the action of the hydrodynamic forces reduces to the moment of the forces. It is then possible to change to the new generalized coordinates using the formula $s_i=(\lambda_{Oi}/\mu_i)\gamma_i$. All the coefficients of Eqs (2.9), in which the hydrodynamic coefficients $\mu_n,\lambda_n,\lambda_{On},\lambda'_{On}(n=1,2,\ldots)$ appear, are expressed in the first case in terms of the masses $m_n=\lambda_n^2/\mu'_n$ and, in the second case, in terms of the moments of inertia $I_i=\lambda_{Oi}^2/\mu_i$ of the equivalent oscillators. These hydrodynamic parameters and the coordinates of the vectors $\mathbf{r}_n$ are determined experimentally.[7] Moreover, the directions of the principal vectors of the hydrodynamic forces and moments, that is, the directions of the vectors λ_n and λ_{On}, must be known.

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